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On the strong chromatic index of cyclic multigraphs

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Abstract

We study the strong chromatic index of multigraphs whose underlying graph is a cycle. In this paper exact values in some cases and a tight general upper bound are given.

1. Introduction

The *strong chromatic index* $\mathrm{sq}(G)$ of a multigraph G is the smallest number of colours needed to colour the edges of G so that each colour class is an induced matching.

Let C stand for a *cyclic multigraph* which consists of a cycle together with a set of copies of edges of the cycle. Then $C = {}^pC_n$ (with $n \geqslant 3$) if C has n vertices and multiplicity of each edge in C is p, $p \in \mathbb{N}$. If $p = 1$ then ${}^1C_n = C_n$, the n-cycle.

This paper studies the strong chromatic index of cyclic multigraphs C. It seems that the first to study the strong chromatic index were Fouquet and Jolivet. In [7,8] they deal with this notion for some cubic graphs. The notion of sq has become popular due to a conjecture of Erdős and Nešetřil [4,3], who ask whether it is true that for a graph G with maximum degree $\Delta(G)$

$$\mathrm{sq}(G) \leqslant \tfrac{5}{4}\Delta(G)^2.$$

Specifically, if $\Delta(G)$ is odd, the question, due to Faudree et al. [6], is if

$$\mathrm{sq}(G) \leqslant \tfrac{5}{4}\Delta(G)^2 - \tfrac{1}{2}\Delta(G) + \tfrac{1}{4}.$$

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So far, the conjecture has been settled [1,10] only when $\Delta(G)\leqslant 3$. Other interesting results concerning the strong chromatic index of graphs can be found in [5,6].

In [9] we study the strong chromatic index for multigraphs. The main results of [9] concerning $\mathrm{sq}(C)$ are summarized in Section 3 below. It turns out that while it is easy to determine the strong chromatic index for a multigraph with a tree structure, the story changes dramatically if the underlying graph contains a cycle. In fact, we are not able to find a general formula providing a strong chromatic index for a cyclic multigraph C, cf. comments in Section 3. In this paper, which is a sequel to [9], we concentrate on exact values of the strong chromatic index for some classes of cyclic multigraphs C (see Theorems 9 and 10) and in Theorem 12 and Proposition 13 we improve on a general upper bound given in [9] (cf. Theorem 6 below). The exact value $\mathrm{sq}(C)$ is found for seemingly simple cyclic multigraphs. Namely, each such C has two consecutive edges both with multiplicity one or there is an edge of multiplicity one among every three consecutive edges of C. In the latter case C is called to be circular-saw-like. On the other hand, circular-saw-like multigraphs C prove useful. It is them which we use to show that each of three expressions of which the new upper bound is built in Theorem 12 is essential. Nevertheless, our proofs are not simple which — as we feel — is due to the difficulty of the problem.

The chromatic index $q(C)$ of any cyclic multigraph C is determined by Berge [2, p. 255]:

$$q(C)=\begin{cases}\Delta(C) & \text{if } n \text{ is even,}\\ \max\left\{\Delta(C),\left\lceil\frac{e(C)}{\lfloor n/2\rfloor}\right\rceil\right\} & \text{otherwise.}\end{cases} \tag{1.1}$$

We note that the Erdős–Nešetřil conjecture is generalized in [14] so that the strong chromatic index is replaced by the distance-d chromatic index, where $d\in\mathbb{N}$, and the graph is replaced by a loopless multigraph G. The conjectured upper bound is a polynomial in $\Delta(G)$ of degree d with coefficients depending on the parity of $\Delta(G)$. The bound coincides with those given above if $d=2$.

The *distance-d chromatic index*, $q^{(d)}(G)$, introduced in Skupień [14], is the smallest cardinality among partitions of the edge set $E(G)$ into d^{+} *matchings* each of which, by definition, is an independent set of dth power $L(G)^{d}$ of the line graph $L(G)$ of G. In other words, a d^{+} matching in G comprises edges at mutual edge distance (which coincides with the distance in the line graph) larger than d. Thus a 1^{+} matching is simply a matching, and 2^{+} matching is an induced matching. Therefore $q^{(1)}=q$, the chromatic index, and $q^{(2)}=sq$, the strong chromatic index. Note that distance vertex-colourings are started by Florica and Horst Kramers [11,12].

The exact order of growth of the distance-d chromatic index (number) of n-dimensional hypercube Q^{n} is established in [14]. In the same paper one can also find a formula for the distance d-chromatic index of $C={}^{p}C_{n}$:

$$q^{(d)}({}^{p}C_{n})=\begin{cases}pn & \text{if } n\leqslant 2d+1,\\ \left\lceil\frac{pn}{\lfloor n(/d+1)\rfloor}\right\rceil & \text{if } n>2d+1\end{cases}$$

which coincides with $q({}^{p}C_{n})$ if $d=1$ and gives $\mathrm{sq}({}^{p}C_{n})$ if $d=2$.

2. Preliminaries

For a multigraph G, the *strong chromatic index* $\mathrm{sq}(G)$ is defined as the smallest k for which the edge set of G admits a *strong colouring* with k colours where, by definition, each colour class is an induced matching. Given a multigraph G, we define the *cluster number* $\eta(G)$ of G so that it coincides with the $d^\wedge$ cluster number $\omega_1^d(G)$ (introduced in [14]) if $d=2$. Namely, $\eta(G)$ is the largest size $e(H)$ among submultigraphs H of G without any induced matching of size two. We call the edge set of such a submultigraph to be a *cluster* in G. Thus, a cluster comprises edges of G so that any two of them coincide or are adjacent either mutually or to a common neighbour in $E(G)$.

Let $C = C(n; a_1, a_2, \ldots, a_n)$ (with $n \geqslant 3$) denote the multigraph obtained from the n-cycle C_n with edges denoted (along the cycle) as $e_1, e_2, \ldots, e_n$ by replacing each e_i with $a_i \geqslant 1$ parallel edges. Let $\Delta(C)$ and $e(C)$ denote the maximum degree and the number of edges in C, respectively. It can be easily seen that the Berge formula (1.1) reduces to

$$q(C) = \max\left\{ \Delta(C), \left\lceil \frac{e(C)}{\lfloor n/2 \rfloor} \right\rceil \right\}. \tag{2.1}$$

In the definition of $C(n; a_1, a_2, \ldots, a_n)$ all a_i have to be positive. However, it turns out that the concept of the strong chromatic index can be naturally extended to the case with some a_i equal to 0. A strong colouring in case any $a_i = 0$ can be viewed as a partial strong colouring which leaves all edges on ith position in C uncoloured. In case of general multigraph we allow for 0-*edges* in the underlying graph so that 0-edges contribute to the distance function only (and are considered non-existing otherwise). We choose to employ the language of sequences rather than multigraphs to incorporate this generalization.

A cyclic sequence A of *length* n ($n \geqslant 3$) is given as $A = (n; a_1, a_2, \ldots, a_n)$, where the $a_i \geqslant 0$ are cyclically ordered. The *size* $e(A)$ of A is then $a_1 + a_2 + \cdots + a_n$. Since we will mostly deal with cyclic sequences we will usually omit the word cyclic and explicitly specify if a sequence is not to be viewed as cyclically ordered.

In what follows each number indicating a position in a sequence is to be read modulo n, with the exception that 0 is to be replaced by n.

We define the *triple* $T_i(A)$ to be the non-cyclic sequence (a_i, a_{i+1}, a_{i+2}), and its *size* $t_i(A)$ to be the number $a_i + a_{i+1} + a_{i+2}$. The *cluster number* of A, $\eta(A)$, is then the maximum of $t_i(A)$ over all i, $1 \leqslant i \leqslant n$, provided that $n \geqslant 7$. Additionally, $\eta(A) = e(A)$ if $n \leqslant 5$. For $n = 6$, $\eta(A)$ is the maximum among numbers $a_1 + a_3 + a_5$, $a_2 + a_4 + a_6$ and all $t_i(A)$, $i = 1, 2, \ldots, 6$.

By a *basic sequence* we mean a sequence whose each term is either 0 or 1, and in which every 1 is followed by at least two zeros. The *strong chromatic index* $\mathrm{sq}(A)$ of a sequence A is then defined as the smallest k such that A decomposes into a sum of k basic sequences.

It should be clear that if, in $A = (n; a_1, a_2, \ldots, a_n)$, we restrict all a_i to positive integers, then A is equivalent to $C(n; a_1, a_2, \ldots, a_n)$. Also, the cluster number and the strong chromatic index as defined for sequences are under this equivalence equal to the respective concepts for the $C(n; a_1, a_2, \ldots, a_n)$.

Given two (cyclic) sequences $A = (n; a_1, a_2 \ldots, a_n)$ and $B = (n; b_1, b_2, \ldots, b_n)$ of the same length n, we say that A *dominates* B (or, equivalently, that B *is dominated by* A) if, for every i, $a_i \geqslant b_i$.

If A dominates B, then the *difference* $A - B := (n; a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$. If A or B is a cyclic multigraph then $A \dot{-} B$ stands for the difference of sequences, each multigraph therein being replaced by the corresponding equivalent sequence.

If $A = (n; a_1, a_2, \ldots, a_n)$, and $a_i \neq 0$ for some i, then the *greedy sequence* $G_i(A) = (n; g_1, g_2, \ldots, g_n)$ is a basic sequence defined by the following rule (we only specify which g_j are equal to 1). Set $g_i = 1$ and $i_0 = i$. Then proceed recursively to define the sequence $\{i_k\}$ of those indices i_k for which $g_{i_k} = 1$: given i_k, let i_{k+1} be the first number in $i_k + 3, i_k + 4, i_k + 5, \ldots,$ for which $a_{i_{k+1}} \neq 0$. Also, let $g_{i_{k+1}} = 1$. Do this as long as the sequence defined this way is basic, that is, the last i_m for which $g_{i_m} = 1$ will be the one for which the set $\{i_{m+1}, i_{m+1} + 1, i_{m+1} + 2\}$ contains i.

3. Useful quotations

The following results up to Theorem 6 appear in our recent paper [9]:

$$\mathrm{sq}(C) \geqslant \max\left\{\eta(C), \left\lceil \frac{e(C)}{\lfloor n/3 \rfloor} \right\rceil\right\} \tag{3.1}$$

$$\mathrm{sq}(C) = \eta(C) \quad \text{for } n \leqslant 6. \tag{3.2}$$

Lemma 1. *Let $A = (n; a_1, a_2, \ldots, a_n)$, and let $e(A) = s\eta(A) + r$, where $s \geqslant 1$ and $0 \leqslant r < \eta(A)$. Then*

$$\mathrm{sq}(A) \leqslant \left\lceil \frac{e(A)}{s} \right\rceil$$

Theorem 2. *Let $A = (n; a_1, a_2, \ldots, a_n)$ be given. Then*

(i) $\mathrm{sq}(A) = \left\lceil \frac{e(A)}{\lfloor n/3 \rfloor} \right\rceil$ *if $e(A) \geqslant \lfloor n/3 \rfloor \eta(A)$, and*

(ii) $\mathrm{sq}(A) = \eta(A)$ *if $e(A) \leqslant \lceil n/5 \rceil \eta(A)$.*

Corollary 3. *Let $n = 7, 8$ or 11, and let $A = (n; a_1, a_2, \ldots, a_n)$. Then*

$$\mathrm{sq}(A) = \max\left\{\eta(A), \left\lceil \frac{e(A)}{\lfloor n/3 \rfloor} \right\rceil\right\}.$$

Proposition 4. *Assume $n = 9$. Let $m_i = \min\{a_i, a_{3+i}, a_{6+i}\}$, $i = 1, 2, 3$. Then*

$$\mathrm{sq}(A) = \max\left\{\eta(A), \left\lceil \frac{e(A) - (m_1 + m_2 + m_3)}{2} \right\rceil\right\}.$$

Lemma 5. *Let* $A=(n;a_1,a_2,\dots,a_n)$, *and assume that* $a_i=a_{i+1}=0$ *for some* i. *Then* $\mathrm{sq}(A)=\eta(A)$.

Theorem 6. *For every sequence* $A=(n;a_1,a_2,\dots,a_n)$ *with* $n>4$,

$$\mathrm{sq}(A)\leqslant\eta(A)+\left\lceil\frac{\eta(A)}{\lceil(n-4)/2\rceil}\right\rceil.$$

It can be shown [13] that parameters like order, size $e(C)$, maximum degree $\Delta(C)$, cluster number $\eta(C)$, minimum degree $\delta(C)$, and also minimum size of a triple in C are not sufficient to derive an exact formula on $\mathrm{sq}(C)$. To this end, let $C=(9;4,4,1,5,2,4,2,4,2)$ and $C'=(9;4,4,1,\ 5,3,3,2,3,3)$ be cyclic multigraphs of order nine. Then $e(C)=e(C')=28$, $\Delta(C)=\Delta(C')=8$, $\eta(C)=\eta(C')=11$, $\delta(C)=\delta(C')=5$, minimum size of a triple is eight in both cases, whereas, according to Proposition 4, $12=\mathrm{sq}(C)\neq\mathrm{sq}(C')=11$.

Moreover, Proposition 4 has the following counterpart for $n=12$.

Proposition 7 (Meszka [13]). *Let* $A=(12;a_1,a_2,\dots,a_{12})$ *and* $m_i=\min\{a_i,a_{3+i},a_{6+i},a_{9+i}\}, i=1,2,3$. *Then*

$$\mathrm{sq}(A)=\max\left\{\eta(A),\ \left\lceil\frac{e(A)-(m_1+m_2+m_3)}{3}\right\rceil\right\}.$$

4. Exact values

We start by a strengthening of the above Theorem 2(ii) for each $n\geqslant32$ and six smaller values of n. Consider the following condition.

(C0) *No two consecutive positions in* A *are filled in with zeros.*

Theorem 8. *For* $n\geqslant4$, $\mathrm{sq}(A)=\eta(A)$ *if* $e(A)\leqslant\lfloor\frac{n}{4}\rfloor\eta(A)$.

Proof. Due to Lemma 5, assume (C0).

Case I: There is a quadruple F in A whose both triples have size less than $\eta(A)$. We can assume that notation is chosen so that $F=F_1(A):=(a_1,a_2,a_3,a_4)$ and $a_5\neq0$.

Consider the greedy sequence $G_5(A)=(n;g_1,\dots,g_n)$. Then $g_5=1$. Suppose $g_k=1$ with $k\leqslant n-2$ such that next 1 in $G_5(A)$ is g_{k+r} with $r>3$. Hence $a_{k+3}=0$ and $r=4$ by (C0). Then the triple $T_{k+1}(A)$, which is "omitted" by $G_5(A)$, has size $t_{k+1}(A)=a_{k+1}+a_{k+2}<\eta(A)$ because otherwise $t_k(A)>\eta(A)$, a contradiction. Hence and by the choice of F, the sequence $G_5(A)$ has 1 as a term in each triple in A of size $\eta(A)$. To see this, we need only to consider the "end" of $G_5(A)$. Then the only doubtful case is that $a_2=0$ and $g_{n-1}=1$. Then, however, $t_n(A)<\eta(A)$ because otherwise $t_{n-1}(A)>\eta(A)$ would be a contradiction. Finally, the size of $G_5(A)$ is at least $\lfloor n/4\rfloor$ due to condition (C0).

Case II: The opposite of Case I holds. Choose notation so that $a_1 \neq 0$. Then any quadruple F is a member of a family of exactly $\lfloor n/4 \rfloor$ disjoint quadruples each of which includes a triple of size $\eta(A)$. Owing to the assumption on $e(A)$, the sequence A has zeros on all positions which do not belong to those triples. In particular, $4|n$ due to (C0). Furthermore, zero and non-zero entries in A alternate. In fact, because $a_1 \neq 0$ in F_1, we see that $a_4 = 0$ whence, by (C0), $a_5 \neq 0$. Therefore, in F_2 the triple of size $\eta(A)$ occupies the last three positions whence $a_2 = 0$. Thus we can see that exactly even-numbered positions are filled in with zeros. Consider the greedy sequence $G_5(A)$. Its size is $n/4$ because $g_k = 1$ in $G_5(A)$ for $k = 1 + 4i,\ i = 0, 1, \ldots, n/4 - 1$.

Thus replacing A by $A - G_5(A)$ in either case gives a sequence which satisfies the assumption of Theorem on $e(A)$. While condition (C0) holds, iterate producing both $G_5(A)$ and a new A. Let A_0 denote the initial sequence A, let x be the number of iteration and let B stand for the resulting sequence A $(=A_x)$. Then

$$\mathrm{sq}(A_0) \leqslant x + \mathrm{sq}(B) = x + \eta(B) = \eta(A_0). \qquad \Box$$

As shown with examples in [9], the upper bound in Theorem 6 can be attained, or at least one can get very close to it, for all values of n and $\eta(C)$. In what follows we determine the strong chromatic index of a relatively large class of cyclic multigraphs which includes the mentioned examples.

We will confine ourselves to the case of cyclic multigraphs $C = C(n; a_1, a_2, \ldots, a_n)$ with all $a_j > 0$. An ith position with $a_i = 1$ and that with $a_i > 1$ are called *a saddle* and *a hill*, respectively. By a *triple* in C we mean the submultigraph induced by all edges on three consecutive positions. A triple is called a 3-*block* if the first two positions are hills. A hill [triple] is called *solitary* if both its neighbouring positions are saddles in C. A saddle which is preceded by a solitary hill is called *a solitary saddle*. Let h and θ be the *number of hills* and *number of solitary hills* in C, respectively. We say that C is *circular-saw-like multigraph* if any two saddles are separated by hills in C and no three hills are consecutive along C. In what follows in this section, C stands for a circular-saw-like multigraph. The set of solitary hills and that of solitary saddles are denoted H' and S', respectively, their common cardinality being clearly θ. Note that the number of 3-blocks in C is $(h - \theta)/2$. Therefore, $n = 2\theta + 3(h - \theta)/2$ whence

$$\theta = 2n - 3h \tag{4.1}$$

and $n/2 \leqslant h \leqslant 2n/3$.

Given any natural number a, let $\mathscr{S}$, $\mathscr{S} = \mathscr{S}(n, \theta; a)$, be the class of circular-saw-like multigraphs C of order n with θ solitary hills and multiplicity exactly a at each hill. Given a cyclic submultigraph C^* of C, let B be a cyclic sequence obtained from the sequence $C \dot{-} C^*$ by removal of all positions which are saddles in C, in symbols,

$$B = \xi(C \dot{-} C^*).$$

In what follows $C^* = C_n$ (a simple cycle) or $C^* \in \mathscr{S}(n, \theta; a)$ for $a \geqslant 2$.

Theorem 9. *Let C be a circular-saw-like multigraph with θ solitary hills and of order $n \geqslant 9$. Then*

$$\mathrm{sq}(C) = \max\left\{\eta(C), \left\lceil \frac{e(C)}{\lfloor n/3 \rfloor} \right\rceil, \left\lceil \frac{e(C)-\theta}{\lfloor h/2 \rfloor} \right\rceil \right\}.$$

Proof. *Case* I: θ is even and $\theta \notin \{2,4\}$. Hence h is even. Then C can be seen as a union of $\theta/2$ saddles and $h/2$ disjoint triples each with two hills and a saddle. Therefore $\lceil (e(C)-\theta)/\lfloor h/2 \rfloor \rceil \leqslant \eta(C)$. Moreover, $\lfloor n/3 \rfloor = h/2 + \lfloor \theta/6 \rfloor$ and clearly $\eta(C) \geqslant 5$. It is immediate that if $\theta = 0$ then $e(C) \leqslant (n/3)\eta(C)$. Else, if even $\theta \geqslant 6$, it is easy to check that $e(C) \leqslant \lfloor n/3 \rfloor \eta(C) - \lfloor \theta/6 \rfloor \eta(C) + \theta/2 \leqslant \lfloor n/3 \rfloor \eta(C)$. Therefore, $\lceil e(C)/\lfloor n/3 \rfloor \rceil \leqslant \eta(C)$. Consequently, by (3.1), it remains to show that

$$\mathrm{sq}(C) \leqslant \eta(C).$$

Consider subcases.

(α): $6 \mid \theta$. Then $3 \mid n$. Let $B = \xi(C \dot{-} C_n)$. Hence the length of B is h and is even. Therefore,

$$\mathrm{sq}(C) \leqslant \mathrm{sq}(C_n) + \mathrm{sq}(C \dot{-} C_n) = 3 + \mathrm{q}(B) = 3 + \Delta(B) = \eta(C).$$

(β): 6 θ. Then $\theta = 8 + 6k$ or $\theta = 10 + 6k$ for some $k \in \mathbb{N}_0$.

Claim 9.1. *For any $C' \in \mathscr{S}(n, \theta; 2)$ with $\theta \geqslant 8$ and $\theta \equiv 2, 4 \pmod 6$, $\mathrm{sq}(C') = 5$.*

Proof. We proceed by induction on n.

(i) If $\theta = 8 = h$ (and $n = 16$), one can see that a recursive greedy procedure gives the result. If $\theta = 10 = h$ (and $n = 20$), a required strong colouring with colours $1, \ldots, 5$ can be as follows.

$$\overset{2}{1}\;\;3\;\;{}_{x}\;\;\overset{5}{4}\;\;1\;\;\overset{3}{2}\;\;4\;\;\overset{5}{1}\;\;2\;\;\overset{4}{3}\;\;5\;\;\overset{2}{1}\;\;3\;\;\overset{5}{4}\;\;1\;\;\overset{3}{2}\;\;4\;\;\overset{5}{1}\;\;2\;\;\overset{4}{3}\;\;5.$$

Note that the initial quadruple therein coincides with that for $n = 16$, x being a special vertex in the tuple. The following colouring in a 16-tuple can be viewed as an extension of that in the initial quadruple in which x is replaced by a suitable 12-tuple of alternating hills and saddles (which proves the result when $\theta = h$, i.e., $n = 2\theta$):

$$\overset{2}{1}\;\;3\;\;\underbrace{\overset{5}{4}\;\;1\;\;\overset{3}{2}\;\;4\;\;\overset{5}{1}\;\;2\;\;\overset{4}{3}\;\;1\;\;\overset{5}{2}\;\;4\;\;\overset{3}{1}\;\;2}_{x}\;\;\overset{5}{4}\;\;1$$

(ii) Consider the case that $h > \theta$ in a C'. Assume that the result is true for a C'' obtained from C' by contracting a 3-block to a vertex x. The following transformation

$$(C'')\quad \overset{2}{1}\;\;3\;\;{}_{x}\;\;\overset{5}{4} \quad\mapsto\quad \overset{2}{1}\;\;3\;\;\underbrace{\overset{5}{4}\;\;\overset{2}{1}\;\;3}_{x}\;\;\overset{5}{4} \quad (C') \tag{4.2}$$

shows how to pass from a strong 5-colouring of $E(C'')$ to that of $E(C')$ by only adding colours to the edges of the restored 3-block. This completes the proof of the Claim 9.1 □

Assume that $C \supseteq C' \in \mathscr{S}(n,\theta;2)$ and let $B=\xi(C\dot{-}C')$. Conclude that

$$\mathrm{sq}(C)\leqslant \mathrm{sq}(C')+\mathrm{sq}(C\dot{-}C')=5+\Delta(B)=\eta(C).$$

Case II: θ is odd, $\theta\neq 1$ and all hills have multiplicity at least a such that $a\geqslant(\theta-1)/2$. Then h is odd and $ah+(h-\theta)/2+\theta\leqslant e(C)$. Hence $(\theta-1)h/2+(h+\theta)/2=\theta(h+1)/2\leqslant e(C)$. From this, $e(C)(h-1)/2\leqslant(e(C)-\theta)((h-1)/2+1)$ and because $\theta\geqslant 3$, $\lfloor n/3\rfloor\geqslant(h+1)/2$. Therefore, $\lceil e(C)/\lfloor n/3\rfloor\rceil\leqslant\lceil(e(C)-\theta)/\lfloor h/2\rfloor\rceil$. Thus it remains to prove

$$\mathrm{sq}(C)=\max\left\{\eta(C),\left\lceil\frac{e(C)-\theta}{\lfloor h/2\rfloor}\right\rceil\right\}.$$

Consider two cases to prove the upper bound first.

(i) $\theta=3$. Hence $3\mid n$. Let $B=\xi(C\dot{-}C_n)$. Then

$$\begin{aligned}\mathrm{sq}(C)&\leqslant \mathrm{sq}(C_n)+\mathrm{sq}(C\dot{-}C_n)=3+\mathrm{q}(B)\\&=3+\max\left\{\Delta(B),\left\lceil\frac{2e(C\dot{-}C_n)}{h-1}\right\rceil\right\}=\max\left\{\eta(C),\left\lceil\frac{e(C)-\theta}{\lfloor h/2\rfloor}\right\rceil\right\}.\end{aligned}$$

(ii) $\theta\geqslant 5$. Let $C'\in\mathscr{S}(n,\theta;(\theta-1)/2)$. Assume that C'' is obtained from C' by contracting each 3-block to a vertex. Then $C''\in\mathscr{S}(2\theta,\theta;(\theta-1)/2)$. Now, a strong colouring of $E(C'')$ with $\eta(C'')=\theta$ colours clearly exists. Namely, each colour class includes one edge from a saddle and an edge from each of $(\theta-1)/2$ greedily chosen hills. As in (4.2) we can locally extend colouring of $E(C'')$ to that of $E(C')$. Thus $\mathrm{sq}(C')\leqslant\theta$ whence, by (3.1), $\mathrm{sq}(C')=\theta$. Assume that $C'\subseteq C$ and let $B=\xi(C\dot{-}C')$. Hence

$$\begin{aligned}\mathrm{sq}(C)&\leqslant \mathrm{sq}(C')+\mathrm{sq}(C\dot{-}C')=\theta+\mathrm{q}(B)\\&=\theta+\max\left\{\Delta(B),\left\lceil\frac{2e(C\dot{-}C')}{h-1}\right\rceil\right\}=\max\left\{\eta(C),\left\lceil\frac{e(C)-\theta}{\lfloor h/2\rfloor}\right\rceil\right\}.\end{aligned}$$

Now we find a lower bound for $\mathrm{sq}(C)$ in order to prove equality in both cases (i) and (ii) above. Let $\tau=1,2,\ldots,t$ be all colours each of which is applied at a solitary saddle. Let k_τ be the number of solitary saddles with colour τ. Then $\sum_{\tau=1}^{t}k_\tau=\theta$. One can see that the colour class of any τ includes at most $(h-k_\tau)/2$ edges of which none is at a solitary saddle. Hence the total number of edges of C associated with t colours τ, denoted e', is

$$e'\leqslant\sum_{\tau=1}^{t}\left(k_\tau+\frac{h-k_\tau}{2}\right)\leqslant\theta+t\,\frac{h-1}{2}.$$

It is easily seen that each remaining colour class includes at most $(h-1)/2$ edges. Therefore, the number of those colour classes is at least $2(e(C)-e')/(h-1)$, whence

$$\mathrm{sq}(C) \geqslant t + \frac{2e(C)-2e'}{h-1} \geqslant \frac{2e(C)-2\theta}{h-1}.$$

Case III: The opposite of Cases I and II holds. Then either

(α): $\theta \in \{1,2,4\}$ or

(β): θ is odd and there is a hill having multiplicity $a \geqslant 2$ such that $a < (\theta-1)/2$ (whence $\theta \geqslant 7$ and $n \geqslant 2\theta \geqslant 14$).

In case (α), 3 n. Moreover, by (4.1), $\lfloor h/2 \rfloor = \lfloor n/3 \rfloor$. In case ($\beta$), $e(C) < \eta(C)$ $(h-1)/2+(\theta-1)/2+(\theta+1)/2$ where the middle summand bounds a and the last one, $(\theta+1)/2$, is the number of saddles which do not contribute to the first summand. Hence $\lceil (e(C)-\theta)/\lfloor h/2 \rfloor \rceil \leqslant \eta(C)$ can be seen in case (β). Therefore, our Theorem reduces to equality in (3.1) whence and by Theorem 2(i), to the inequality $\mathrm{sq}(C) \leqslant \eta(C)$ under the assumption

$$\eta(C)\left\lfloor \frac{n}{3} \right\rfloor > e(C). \tag{4.3}$$

Consider two main subcases.

Case III.1: $\theta \geqslant 9$ in case (β). Then $n \geqslant 2\theta \geqslant 18$. Let C' be a submultigraph of C such that $C' \in \mathscr{S}(n,\theta;a)$. Assume that C^* is obtained from C' by contracting each 3-block to a vertex. Then $C^* \in \mathscr{S}(2\theta,\theta;a)$. Let $l{:=}2a+1$ $(<\theta)$ and let $C'' \in \mathscr{S}(2l,l;a)$. Hence $\eta(C'') = l \geqslant 5$ and a strong colouring of $E(C'')$ with $\eta(C'')$ colours exists, cf. Case II(ii).

Strong colouring of $E(C^*)$ with l colours can be obtained from that of $E(C'')$. Restore C^* from C'' by inserting a number of short sequences of alternating hills and saddles of lengths denoted by k. Let $a \geqslant 3$. Firstly, take $k=2r$ if $r{:=}((\theta-l)\mathrm{mod}6) \in \{2,4\}$. Else $r=0$ because $\theta-l$ is even. Next, insert $(\theta-l-r)/6$ such sequences of length $k=12$ each. Adding colours to new edges which is accompanied by local recolouring in first two cases ($k=4$, 8) is presented in Fig. 1, where the symbol x stands for a vertex replaced by a sequence. Recolouring consists in changing a colour on each position indicated by an arrow in Fig. 1. Capital Greek letters (Γ and Λ) represent sets of colours, single colours are denoted by natural numbers.

In the remaining case, $a=2$, presented in Fig. 1, replace the vertex x of C'' by a sequence of length $k=8$ to get C^* of length 18. To get C^* of length 26 replace both x and y. Strong colourings in those C^* are obtained by adding colours to new edges and replacing colour 2 by 5 and possibly the colour 3 by 1 in the saddle immediately following the inserted sequence (which replaces x and possibly y), cf. arrows in Fig. 1 ($a=2$). C^* with any larger θ is obtainable by inserting sequences of length $k=12$ each to one sequence of length 10, 18, or 26.

Adding colours if $k=12$ for any $a \geqslant 2$ does not require any recolouring, see Fig. 1.

As in Case II(ii) we locally extend strong colouring of $E(C^*)$ to that of $E(C')$ using l colours only. Furthermore, by Lemma 5, $\mathrm{sq}(C \dot{-} C') = \eta(C \dot{-} C') = \eta(C) - l$. Therefore,

$$\mathrm{sq}(C) \leqslant \mathrm{sq}(C') + \mathrm{sq}(C \dot{-} C') = \eta(C).$$

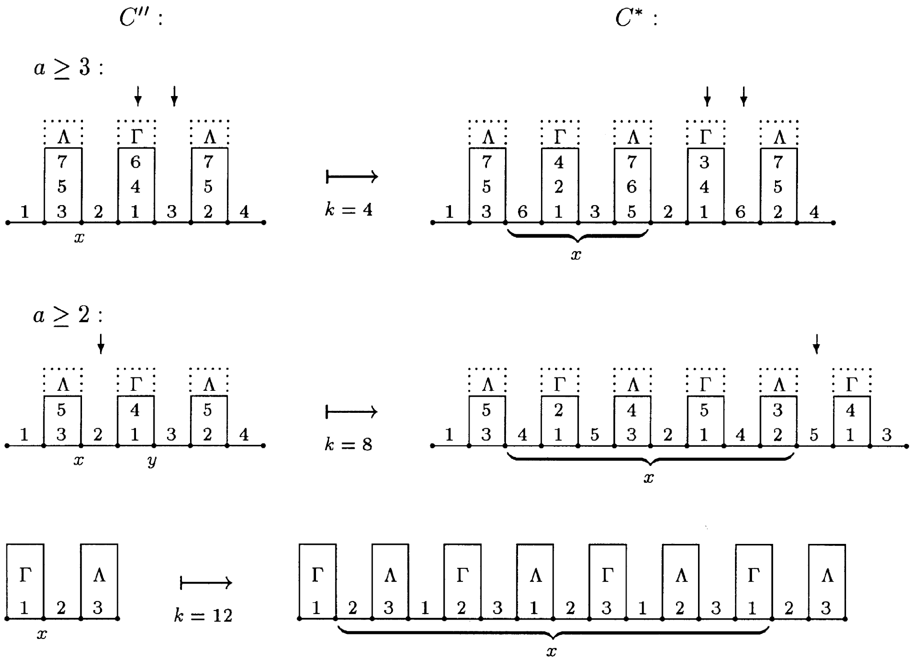

Fig. 1.

Case III.2: Either (α) holds or $\theta = 7$ (and $a = 2$) in case (β). Proceed to transform C into another circular-saw-like multigraph C^* (whose order we denote m) by recursively applying the following operation. Contract to a vertex any non-solitary triple of size $\eta(C)$. Then $\eta(C^*)$ does not increase. Repeat this contracting as long as possible. Keep track of this process so that going back from C^* to C could be possible. Then the number of solitary hills in C^* remains θ. Moreover, $\eta(C^*) \leqslant \eta(C)$. From (4.3) one can get

$$\eta(C) \left\lfloor \frac{m}{3} \right\rfloor > e(C^*) \tag{4.4}$$

whence $m \geqslant \max\{2\theta, 4\}$.

Claim 9.2. *Suppose*

$$\mathrm{sq}(C^*) \leqslant \eta(C). \tag{4.5}$$

Then Theorem 9 *holds.*

Proof. Apply the converse operation in order to transform C^* back to C. At each step, new C^* is obtained by expanding a vertex, x, of the old C^* to the corresponding triple. Now, it is enough to show that no step of the process spoils the inequality (4.5). To this end, assume that $\eta(C)$ colours are fixed and some (or all) of them are used in old C^*. Keep those colours in old C^* unchanged and add a colour to each edge of the restored triple as sketched in Fig. 2.

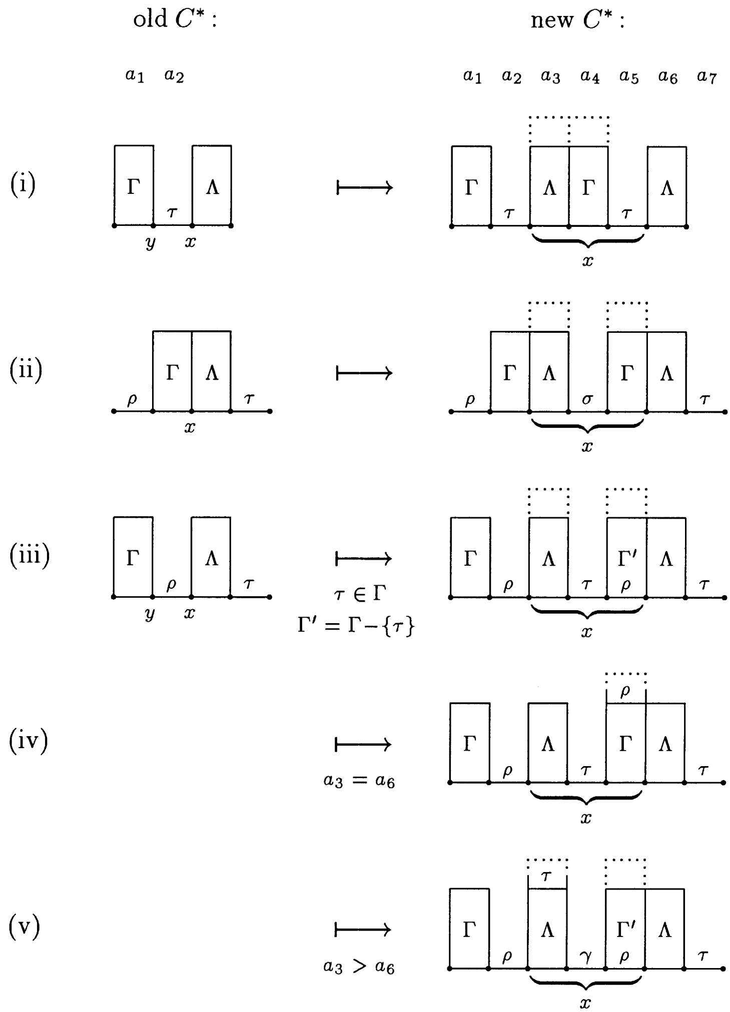

Fig. 2. Adding colours.

In Fig. 2, sets of colours and single colours are denoted by upper and lower-case Greek letters, respectively. Colours are assumed to give a strong colouring in old C^* whence, e.g., $\Gamma \cap \Lambda = \emptyset$. Furthermore, $\rho = \tau$ is possible only in Fig. 2(ii) and then $\sigma = \rho = \tau$. Else in Fig. 2(ii), $\rho \neq \tau$. Because the size of the restored triple is $a_3 + 1 + a_5 = \eta(C) \geqslant |\Lambda| + |\Gamma| + 2$, either $a_3 > |\Lambda| = a_6$ and then the colour σ in Fig. 2(ii) equals ρ and τ is applied together with Λ at $3rd$ position or else $\sigma = \tau$ and ρ together with Γ is applied at $5th$ position (since $a_5 > a_2$). On the other hand, $\tau \in \Gamma$ is true only in Fig. 2(iii). Additionally, $\gamma \in \Gamma$ and $\Gamma' := \Gamma - \{\gamma\}$ in Fig. 2(v). Dotted lines denote possible edges which can be associated with additional colours taken from among available $\eta(C)$ colours.

The case that y is expanded instead of x in Fig. 2 can be seen to be represented in Fig. 2 (in all subcases but (ii)). Namely, consider expanding x but "read" both sides of the arrow $\mapsto$ from left to right to see a required distribution of colours. This completes the proof of the Claim 9.2. $\square$

We are going to show (4.5):

Subcase (α): Then $4 \leqslant m \equiv 2\theta \pmod 3$ whence $3\,m$. If $m = 4$ or 5 then (4.5) holds because $\mathrm{sq}(C^*) = \eta(C^*) = e(C^*) < \eta(C)$ by (4.4). Analogously, for $m = 7, 8$, or 11, due to Corollary 3 with C and n replaced by C^* and m, respectively, and due to the inequalities $\eta(C^*) \leqslant \eta(C)$ and (4.4), we can see that (4.5) holds.

Assume that $m \geqslant 10$ if $\theta = 2$, else $m \geqslant 14$. Let the simple cycle C_m be a fixed subgraph of C^*. Because m is large, C^* has $(m - 2\theta)/3 \geqslant 2$ disjoint 3-blocks each of which is of size less than $\eta(C)$. Let e, f be two distinct edges of C_m each on the first hill of a 3-block in C^*. Let $C^{**} = C^* \dot{-} (C_m - \{e, f\})$. Then $\eta(C^{**}) \leqslant \eta(C) - 3$ can be seen. If $\theta = 2, 4$ then $\xi(C^{**})$ has even length, whence $\mathrm{q}(\xi(C^{**})) = \eta(C^{**})$ by Berge's result. Else, for $\theta = 1$, $\xi(C^{**})$ is of odd length $2\lfloor m/3 \rfloor + 1$. Therefore, $\lceil e(\xi(C^{**}))/\lfloor m/3 \rfloor \rceil = \lceil (e(C^*) - 3\lfloor m/3 \rfloor)/\lfloor m/3 \rfloor \rceil \leqslant \eta(C) - 3$ by (4.4). Hence, by (2.1), $\mathrm{q}(\xi(C^{**})) \leqslant \eta(C) - 3$ for each θ in our case. We conclude that

$$\begin{aligned} \mathrm{sq}(C^*) &\leqslant \mathrm{sq}(C_m - \{e, f\}) + \mathrm{sq}(C^{**}) \\ &= 3 + \mathrm{q}(\xi(C^{**})) \leqslant \eta(C). \end{aligned}$$

Subcase (β): such that $\theta = 7$. Then $a = 2$. We are going to find and fix an edge e in a saddle of C^* such that e does not belong to any triple in C^* of size $\eta(C)$. To this end, assume $m > 2\theta = 14$. Then C^* has two edges, say e_1 and e_2, both in saddles, which separate two neighbouring hills from the rest of C^*. Put $e = e_1$ if no hill adjacent to e_1 has multiplicity a, else $e = e_2$. Then each triple containing the edge e is non-solitary in C^* and therefore of size less than $\eta(C)$. Moreover, C^* has a saddle whose edge is different from e and adjacent to a hill with multiplicity a. The same requirement can be satisfied if m attains its minimum value $m = 14$. Namely, then all hills are solitary and there is a hill with multiplicity larger then a so that $\eta(C) > 5$ because otherwise (4.4) is false. Moreover, supposing that each saddle belongs to a triple of size $\eta(C)$ in C^* leads to a contradiction too. Let e be the edge in a saddle of a solitary triple of size less than $\eta(C)$.

Let C' be a submultigraph of C^* such that $C' \in \mathscr{S}(m, \theta; 2)$. Then, by the choice of the edge e,

$$\mathrm{sq}(C^* \dot{-} (C' - e)) = \eta(C^* \dot{-} (C' - e)) \leqslant \eta(C) - 5.$$

Let C'' be obtained from C' as follows. Contract one by one all 3-blocks which are non-adjacent to e. Contract next the triple following e to a vertex y if the triple is a 3-block. Contract also the triple preceding e (to a vertex x) if two neighbouring hills precede e and, finally, assume that e is the 0-edge in C''.

Consider the following strong colouring of $E(C'')$

$$\begin{array}{cccccccccccccc}
2 & & 5 & & 3 & & 5 & & 4 & & 5 & & 4 & \\
1 & 3 & 4 & 1 & 2 & 4 & 1 & 2 & 3 & 1 & 2_{x} & e_{y} & 3 & 5,
\end{array}$$

where x and y are possibly special vertices defined above. Adding colours to the edges which possibly replace the vertices x and y is presented below.

$$\begin{array}{cccccccccccccc}
 & 5 & & 4 & 5 & & 4 & 5 & & 4 & 5 & & 4 & \\
1 & 2 & \multicolumn{3}{c}{\underbrace{3 \quad 1 \quad 2}_{x}} & e & 3 & 2 & e & \multicolumn{3}{c}{\underbrace{3 \quad 2 \quad 1}_{y}} & 3 & 5.
\end{array}$$

Adding colours to edges of remaining contracted 3-blocks follows the scheme in (4.2). Conclude that

$$\mathrm{sq}(C^*) \leqslant \mathrm{sq}(C' - e) + \mathrm{sq}(C^* \dot{-} (C' - e)) = 5 + \eta(C^* \dot{-} (C' - e)) \leqslant \eta(C). \qquad \square$$

5. Further exact value

The next result presents a sufficient condition for sq to attain its lower bound (3.1).

Theorem 10. *Let $C = C(n; a_1, a_2, \ldots, a_n)$ be a cyclic multigraph without 0-edges and with multiplicity* 1 *at some two consecutive positions. Then*

$$\mathrm{sq}(C) = \max\{\eta(C), \lceil e(C)/\lfloor n/3 \rfloor \rceil\}.$$

Proof. Assume $e(C) < \lfloor n/3 \rfloor \eta(C)$ because otherwise the result holds by Theorem 2(i). Thus it is enough to show that $\mathrm{sq}(C) \leqslant \eta(C)$ and only for $n \geqslant 9$ due to Corollary 3.

Case I: $3 \mid n$. Then

$$\begin{aligned} \mathrm{sq}(C) &\leqslant 3 + \mathrm{sq}(C \dot{-} C_n) = 3 + \eta(C \dot{-} C_n) \quad \text{by Lemma 5} \\ &= \eta(C). \end{aligned}$$

Case II: $3\,n$, $n \geqslant 10$. Assume that notation is chosen so that $a_4 = 1 = a_5$ in each C under consideration. Let C^0 stand for the initial multigraph C. Let $j = 1, 2, \ldots, 6$ throughout this proof. Perform the following operation. While $n \geqslant 10$, choose a triple $T_j(C)$ of size $\eta(C^0)$ and contract $T_j(C)$ to a vertex, say x. Let C and n stand for the resulting multigraph and its length, respectively. Perform the operation as long as possible in each of the following successive cases only:

(i) $j = 3$, 4; next

(ii) $j = 2$, 5; and finally

(iii) $j = 1$, 6.

One can see that each new C satisfies assumptions of the theorem, $3\,n$ and $e(C) < \lfloor n/3 \rfloor \eta(C^0)$. Furthermore $\eta(C) \leqslant \eta(C^0)$ can also be seen. Moreover, while updating the numbering of positions in each new C, one can assume that at least two out of eight

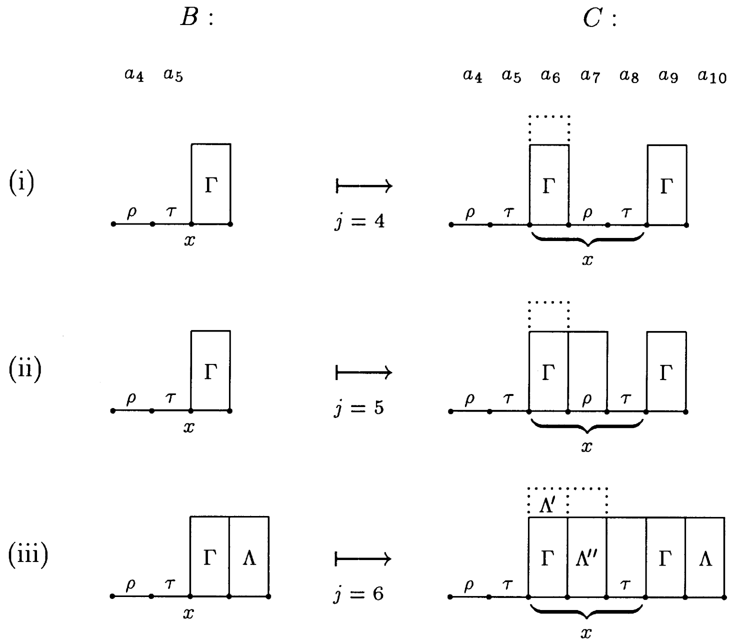

Fig. 3.

positions $1, 2, \ldots, 8$ are to retain their numbering. Let B be the eventual multigraph C. Then either B has length $n = 7, 8$ or all $T_j(B)$ are of size less than $\eta(C^0)$. Note that $e(B) < \lfloor n/3 \rfloor \eta(C^0)$ holds. Hence $\mathrm{sq}(B) \leqslant \eta(C^0)$. In fact, this follows from Corollary 3 if $n = 7, 8$. Else, let C_n be a fixed spanning subgraph of B and let $C' = C_n - \{e, f\}$ where e and f are edges of C_n on positions 3 and 6, respectively. Then C' is a cycle with two 0-edges and such that clearly $\mathrm{sq}(C') = 3$. Moreover, the sequence $B \dot{-} C'$ has two consecutive zeros whence

$$\begin{aligned} \mathrm{sq}(B) \leqslant 3 + \mathrm{sq}(B \dot{-} C') &= 3 + \eta(B \dot{-} C') \quad \text{by Lemma 5} \\ &= \eta(B) \leqslant \eta(C^0). \end{aligned}$$

Assume a strong colouring of $E(B)$ with $\eta(C^0)$ or less colours is given. Let C denote the resulting multigraph at each step of applying the converse operation to the given multigraph which we denote B. Each step consists in inserting a triple in place of a vertex of B possibly in stages (iii), (ii) and (i) taken in this reversed order. Using Fig. 3, we are going to show that adding available colours to newly inserted edges can give a strong colouring of E(C) in each (new) C. Due to symmetry, only one value of j is considered in Fig. 3 at each stage. As in preceding proofs, Greek letters stand for colours (lower case for singles, upper case for sets). Dotted lines mark possibly bigger multiplicities.

At stage (iii), $j = 6$, that is, $t_j(C)$ with $j \neq 1, 6$ are all small and $t_6(C) = \eta(C^0)$. Hence $a_6 \geqslant a_9$ and $a_6 + a_7 \geqslant a_9 + a_{10}$ because otherwise $t_7(C)$ or $t_8(C)$ would be greater than $\eta(C^0)$. Moreover, $a_8 > 1$ because otherwise $t_5(C) = \eta(C^0)$ which is impossible in stage (iii). Adding colours at colliding positions is presented in Fig. 3(iii), where the colour set $\Lambda = \Lambda' \cup \Lambda''$ and $\rho \in \Lambda''$ or else ρ is applied at position 8 in C ($a_8 > 1$).

In Fig. 3(ii), the stage (ii) is presented with $j=5$, i.e., $t_3(C)$ and $t_4(C)$ are both small and $t_5(C) = \eta(C^0)$. Hence $a_5 = 1 = a_8$ and $a_6 \geqslant a_9$ because otherwise $t_6(C)$ or $t_7(C)$ could be too large (bigger than $\eta(C^0)$). Moreover, $a_7 > a_4 = 1$ because $t_4(C) < t_5(C)$ in stage (ii). Analogously, in Fig. 3(i) we clearly have $a_7 = 1 = a_8$. The number $\eta(C^0)$ of available colours is easily seen to be sufficient in each case. □

6. Concluding remarks

The above Lemma 1 (proved in [9]) is clearly equivalent to the following.

Lemma 11. *For every sequence* $A = (n; a_1, a_2, \ldots, a_n)$,

$$\mathrm{sq}(A) \leqslant \left\lceil \frac{e(A)}{\lfloor e(A)/\eta(A) \rfloor} \right\rceil .$$

Theorem 12. *For every sequence* $A = (n; a_1, a_2, \ldots, a_n)$ *with* $n \geqslant 5$,

$$\mathrm{sq}(A) \begin{cases} = \eta(A) & \text{if } e(A) \leqslant \eta(A) \max\left\{ \left\lfloor \frac{n}{4} \right\rfloor, \left\lceil \frac{n}{5} \right\rceil \right\}, \\ \leqslant \min\left\{ \eta(A) + \left\lceil \frac{\eta(A)}{\lceil (n-4)/2 \rceil} \right\rceil \cdot \left\lceil \frac{e(A)}{\lfloor e(A)/\eta(A) \rfloor} \right\rceil \right\} & \text{otherwise.} \end{cases}$$

Proof. The result follows from Theorem 2(ii) and Theorem 8 on the one hand and from Theorem 6 and Lemma 11 on the other. □

Because $\mathrm{sq}(A) \geqslant \eta(A)$, for $n \geqslant 10$ and $n \neq 11$, Theorem 12 is equivalent to the following.

Proposition 13. *For* $n = 10$ *or* $n \geqslant 12$,

$$\mathrm{sq}(A) \leqslant \max\left\{ \eta(A), \min\left\{ \eta(A) + \left\lceil \frac{\eta(A)}{\lceil (n-4)/2 \rceil} \right\rceil, \left\lceil \frac{e(A)}{\lfloor n/4 \rfloor} \right\rceil, \left\lceil \frac{e(A)}{\lfloor e(A)/\eta(A) \rfloor} \right\rceil \right\} \right\}.$$

Examples. We are going to provide examples of circular-saw-like multigraphs C whose strong chromatic indexes are larger than $\eta(C)$ and attain exactly one of bounds of Theorem 6 or Lemma 11 (independently one from another). All these C's have the same multiplicity a at each hill, $a \geqslant 2$. Only the bound in Lemma 11 is attained if, for any integer $k \geqslant 6$, the multigraph C has parameters $n = 3k + 6$, $h = 2k + 3$ (whence $\theta = 3$) and $a = 2k - 2$. Only the bound in Theorem 6 is attained if, for any integer $k \geqslant 2$, the multigraph C has parameters $n = 7k + 5$, $h = 4k + 3$ (whence $\theta = 2k + 1$) and $a = k + 1$.

The upper bound in Theorem 12 can be improved by incorporating case covered by Lemma 5 and/or that by Theorem 10. Another possible version can pertain to cyclic multigraphs only (possibly with solitary 0-edges).

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